Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

RADIATIONLESS ORBITS

By Edwin Bidwell Wilson

Department of Physics, Massachusetts Institute of Technology

Read before the Academy, November 10, 1919

The reaction on an electric charge due to non-uniform motion has been calculated (by Lorentz, for example) and found to be

$$\mathbf{F} = \frac{e^2}{6\pi c^3}\frac{d^2\mathbf{v}}{dt^2} \tag{1}$$

this term **F** is merely the first of a series which converges with extreme rapidity provided the dimensions of the charge are small compared with the distance light would travel in the time required for a sensible charge in the velocity.[1] The activity of the force **F** determines the rate R of radiation of mechanical energy as

$$-R = \mathbf{F}\cdot\mathbf{v} = \frac{e^2}{6\pi c^3}\,\mathbf{v}\cdot\frac{d^2\mathbf{v}}{dt^2} \tag{2}$$

What plane orbits are such as to make the radiation as determined by (2) identically zero? The answer to this question is contained in the general integral of the quadratic differential equation of the third order

$$\mathbf{v}.\frac{d^2\mathbf{v}}{dt^2} = \frac{dx}{dt}\frac{d^3x}{dt^3} + \frac{dy}{dt}\frac{d^3y}{dt^3} = 0. \tag{3}$$

The integration may be obtained easily by considering **v** as the radius vector in the hydrograph. Then $d^2\mathbf{v}/dt^2$ is the acceleration of the moving point in the hydrograph and its radial and tangential components are

$$\frac{d^2v}{dt^2} - v\left(\frac{d\varphi}{dt}\right)^2 \quad \text{and} \quad \frac{1}{v}\frac{d}{dt}\left(v^2\frac{d\varphi}{dt}\right)$$

of φ be the inclination of **v** to a fixed direction. The condition of perpendicularity (3) may thus be written

$$\frac{d^2v}{dt^2} - v\left(\frac{d\varphi}{dt}\right)^2 = 0, \qquad \text{or} \qquad \frac{d\varphi}{dt} = \sqrt{\frac{v''}{v}}.$$

Then

$$\begin{aligned} x &= \int \left[v \cos \int \sqrt{v''/v}\, dt\right] dt, \\ y &= \int \left[v \sin \int \sqrt{v''/v}\, dt\right] dt \end{aligned} \tag{4}$$

are the parametric equations of the path, v being any function of the time. The equations contain three constants of integration, which allow an arbitrary choice of axes, and one arbitrary function v. The accelerations along and perpendicular to the path are v' and $\sqrt{vv''}$; the total acceleration is $\sqrt{v'^2 + vv''}$, and its inclination to the path, $\tan^{-1}(\sqrt{vv''}/v')$.

A simple case may be had by taking $v = e^{-at}$, which represents a particle combing to rest. Equations (4) may be written

$$x = \int_{\infty}^{t} e^{-at} \cos at\, dt, \qquad y = \int_{\infty}^{t} e^{-at} \sin at\, dt.$$

Or

$$x = \frac{1}{\sqrt{2}\, a}\, e^{-at} \cos\left(at - \tfrac{3}{4}\pi\right), \qquad y = \frac{1}{\sqrt{2}\, a}\, e^{-at} \sin\left(at - \tfrac{3}{4}\pi\right).$$

And if

$$\theta = at - 3\pi/4, \qquad r = \frac{1}{\sqrt{2}\, a}\, e^{-\theta - 3\pi/4}$$

is the polar equation of the locus—showing an equiangular spiral with 45° between radius and tangent. If $z = x + iy$,

$$z = \frac{1}{\sqrt{2}\, a}\, e^{-a(1-i)t - 3\pi i/4} \tag{5}$$

The vector velocity dz/dt is 135° ahead of the radius z, and the acceleration is 270° ahead, which means a retarding acceleration decreasing the areal velocity and perpendicular to the radius. This type of acceleration is found in the straight-bore centrifugal gun.

In (5) the magnitudes of the radius $\mathbf{r}$, velocity $\mathbf{v}$, acceleration $\mathbf{v}'$, and its rate $\mathbf{v}''$ are in geometric progression with the ratios $\sqrt{2}\, a$, the value of $\mathbf{v}'$ being $\sqrt{2}\, ae^{-at}$. If this acceleration be resolved along the radius and along the normal to the path, the respective components are $2\sqrt{2}ae^{-at}$ outward and $2\, ae^{-at}$ inward. A constant magnetic field perpendicular to the plane of motion is competent to furnish a component acceleration along the normal to the path and proportional to the velocity. The electric field in a plane perpendicular to the line joining two like charges at its middle point will act radially from the point of equilibrium upon a like charge with a force proportional to the distance. A proper slight adjustment of the intensity of the magnetic field will take care of the force (1). It is therefore possible easily to set up an electrical problem which is satisfied by the path (5). The general dynamical equation for the motion thus adjusted is

$$\frac{d^2z}{dt^2} - 2a\,(1 + aA)\,i\,\frac{dz}{dt} - 2a^2z = A\,\frac{d^3z}{dt^3}, \tag{6}$$

and has the roots $a(1 - i)$ and $1/A$ in addition to $-a(1 - i)$; two of the roots therefore give increasing motions. The root $1/A$ is the large extraneous root that comes in when the force (1) is used;[2] the root $a(1 - i)$ gives a non-radiating orbit. Any orbit of the whole set

$$z = C_1\,e^{-a(1-i)t} + C_2\,e^{a(1-i)t} \tag{7}$$

is also non-radiating.

The orbits (4) have been described as non-radiating because the force (1), being normal to the path, does no work. It is only in this sense that they are free from radiation: the equation of energy holds, the sum of the kinetic and potential energies is constant. (As the motions are supposed to be small, quasi-stationary, relativity effects have been ignored.) From the point of view of electro-magnetic theory the rate of radiation varies as the square of the acceleration and can never vanish in accelerated motion; and there is a coördinate radiation of momentum. The relation between the two rates of radiation is shown by the equation

$$\frac{d}{dt}\left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right) = \frac{d\mathbf{v}}{dt}\cdot\frac{d\mathbf{v}}{dt} + \mathbf{v}\cdot\frac{d^2\mathbf{v}}{dt^2}$$

The total radiation is the same according to both points of view whenever estimated between two instants for which $\mathbf{v}.d\mathbf{v}/dt$ has the same values; but the instantaneous rates cannot be indentically equal except for orbits in which $\mathbf{v}.d\mathbf{v}/dt$ is always the same constant independent of the time. I have discussed elsewhere the effect of replacing the law (1) by its alternative in the case of the rectilinear oscillation,[3]—which then ceases to be an oscillation.

From the point of view of relativity the rate of radiation of energy (and momentum) has been treated by several authors and has been shown to vary with the square of the generalized curvature of the path when regarded as a space-time locus in four dimensions.[4] For any real motion, where the velocity is less than that of light, the radiation must be positive and radiationless orbits, other than straight lines uniformly described, are impossible. If, however, velocities greater than light were possible and if the formula for the rate of radiation still held, the orbits would be those for which the curvature vector was a minimum line. Although this state of affairs may have no present physical interest, it does have a purely mathematical interest, and I shall therefore determine also these orbits.

The expression which must be integrated is [4]

$$\frac{d\mathbf{v}}{dt}\cdot\frac{d\mathbf{v}}{dt} = \left(\mathbf{v}\times\frac{d\mathbf{v}}{dt}\right)\cdot\left(\mathbf{v}\times\frac{d\mathbf{v}}{dt}\right), \tag{8}$$

provided the units are so chosen that the velocity of light is 1. Let R be the radius of curvature of the orbit (in ordinary space). The tangential and normal resolution of acceleration throws (8) into

$$\left(\frac{dv}{dt}\right)^2 + \frac{v^4}{R^2} = \frac{v^6}{R^2} \quad\text{or}\quad \left(\frac{dv}{ds}\right)^2 = \frac{v^4 - v^2}{R^2}.$$

Hence

$$v = \sec\int\frac{ds}{R}. \tag{9}$$

If then any space curve be given intrinsically by the equation $R = f(s)$, equation (9) determines the velocity at which the curve must be traced if there is to be no radiation as estimated by the usual electro-magnetic formula.

---

[1] See H. A. Lorentz, *Theory of Electrons*, Art. 37, B. G. Teubner, Leipzig, 1916.

[2] See M. Planck, *Theorie der Wármestrahlung*, p. 110, J. A. Barth, Leipzig, 1906.

[3] Wilson, E. B., *Boston, Proc. Amer. Acad. Arts Sci.*, **50**, 1914, (105–128).

[4] See Wilson and Lewis, *Ibid.*, **48**, 1912, (387–507), especially p. 481.

THE EFFECT UPON THE ATOM OF THE PASSAGE OF AN ALPHA RAY THROUGH IT[1]

By R. A. Millikan, V. H. Gottschalk and M. J. Kelly

Ryerson Physical Laboratory, University of Chicago

Read before the Academy, November 11, 1919

In 1910, by catching at the instant of ionization the positive residues of atoms ionized by X-rays, and by beta and gamma rays of radium, it was conclusively shown[2] that the act of ionization by these agencies uniformly consists in the detachment of a single negative electron from a neutral atom.

The method consisted in balancing the force of gravity acting upon a minute oil-drop by a strong vertical electrical field, holding the oil-drop under observation in a telescope with the aid of a powerful beam of light,